Remarks on the Schrödinger Representation and Quantization

Presented at BHSU REU Poster Session 2025.

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What is Quantization?

In classical or quantum mechanics, an *observable* is anything that can be measured in an experiment. The same observables can be measured in either paradigm, and quantization is how the two are related: For a certain type of topological space X, a *quantization* of the space is a Hilbert Space $\mathcal{H}$ and a certain type of linear map O from $C_0(X)$, the space of classical observables (continuous functions from X to $\mathbb{R}$), to $\mathcal{B}(\mathcal{H})$, the space of quantum observables. There are two approaches to quantization: geometric quantization and algebraic quantization. Geometric quantization requires a type of differential geometry on $C_0(X)$, but algebraic quantization requires only that $C_0(X)$ be a C^* algebra (a space with addition, not-necessarily-commutative multiplication, the distributive property, and *, a conjugation-like operation, if nontrivial) and is an area of active research.

Representation Theory and the Heisenberg Group

The Heisenberg group, $\text{Heis}(\mathbb{R}^{2n})$, is a Lie group that can be represented by the set of vectors $\begin{bmatrix} v \ t \end{bmatrix}$, for $v = \begin{bmatrix} p \ q \end{bmatrix} \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$, with ordinary addition but multiplication given by $\begin{bmatrix} v_2 \ t_2 \end{bmatrix} \cdot \begin{bmatrix} v_2 \ t_2 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \ t_1 + t_2 + \omega(v_1, v_2) \end{bmatrix}$ for a given antisymmetric bilinear two-form ω, with the commutator as its Lie bracket. The *Schrödinger representation* of index m, π_S^m, maps from $\text{Heis}(\mathbb{R}^{2n})$ to $\text{Aut}(L^2(\mathbb{R}^n))$. It is given by $\begin{bmatrix} p \ q \ s \end{bmatrix} \mapsto$ the operator sending $f(x)$ to $e^{2\pi i m(s+(2x+p)\cdot q)} f(x+p)$. For $n = 1$, $P = \begin{bmatrix} 1 \ 0 \ 0 \end{bmatrix}$ has $(d\pi_S^m(P))(f(x)) = \frac{d}{dt}(\pi_S^m(e^{Pt}))f(x)|_{t=0} = \frac{d}{dx}f(x)$, $Q = \begin{bmatrix} 0 \ 1 \ 0 \end{bmatrix}$ has $(d\pi_S^m(Q))(f(x)) = 4\pi mix f(x)$, and $R = \begin{bmatrix} 0 \ 0 \ 1 \end{bmatrix}$ has $(d\pi_S^m(Q))(f(x)) = 2\pi mi f(x)$, so $[d\pi_S^m(Q), d\pi_S^m(P)] = -4\pi im$ and the Lie algebra given by $d\pi_S^m(\text{Heis}(\mathbb{R}^{2n}))$ mimics quantum operators.

We generalized this representation via what is obtained solely from the system of differential equations $\frac{d}{dt}(\pi_S^m(e^{Pt}))f(x)|_{t=0} = \frac{d}{dx}f(x)$, $\frac{d}{dt}(\pi_S^m(e^{Qt}))f(x)|_{t=0} = 4\pi mix f(x)$, and $\frac{d}{dt}(\pi_S^m(e^{Rt}))f(x)|_{t=0} = 2\pi mi f(x)$, which is any function of the form $2\pi mit_3 f(x + K^3_{\geq 2}[t_1, t_2, t_3] + \mathcal{O}(t_1, t_3)) + \pi mixt_2 f(x + K^2_{\geq 2}[t_1, t_2, t_3] + \mathcal{O}(t_2, t_3)) + f(x + t_1 + K^1_{\geq 2}[t_1, t_2, t_3]) + K^0_{\geq 2}[t_1, t_2, t_3] f(x + t_1) + C(x)$, with $\mathcal{O}(t_i, t_j)$ bilinear and $K^i_{\geq 2}[t_1, t_2, t_3]|_{\vec{0}} = 0$ having no linear or constant component in t_1, t_2, or t_3. Setting all $f(x + \mathcal{O}(t_1, t_j))$ terms to $f(x + t_1)$ and $C(x)$ to zero, $(\pi_S^m(\begin{bmatrix} t_1 \ t_2 \ t_3 \end{bmatrix})f)(x) = (1 + 2\pi mi(t_3 + 2xt_2) + K^0_{\geq 2}[t_1, t_2, t_3])f(x + t_1)$, so $K^0_{\geq 2}[t_1, t_2, t_3] f(x + t_1)$ must remain an L^2 function for arbitrary $K^0_{\geq 2}[t_1, t_2, t_3]$, requiring f to be restricted to an element of the Schwartz space $S(\mathbb{R})$, a dense subset of $L^2(\mathbb{R})$. This is not necessarily unitary, evident from taking $K^0_{\geq 2}[t_1, t_2, t_3] = 0$ to be zero. The (unitary) Schrödinger representation corresponds to taking $C(x) = \mathcal{O}(t_1, t_3) = \mathcal{O}(t_1, t_2) = K^i_{\geq 2}[t_1, t_2, t_3] = 0$ for all $i \in \{1, 2, 3\}$ and $K^0_{\geq 2}[t_1, t_2, t_3] = 2\pi mit_1 t_2 + \sum\limits_{k \geq 2} \frac{1}{k!}(2\pi mi(t_3 + (2x + t_1)t_2))^k$, yielding the expression $(1 + 2\pi mi(t_3 + (2x + t_1)t_2) + \sum\limits_{k \geq 2} \frac{1}{k!}(2\pi mi(t_3 + (2x + t_1)t_2))^k) f(x + t_1) = \sum\limits_{k \geq 0} \frac{1}{k!}(2\pi mi(t_3 + (2x + t_1)t_2))^k f(x + t_1) = e^{2\pi mi(t_3 + (2x + t_1)t_2)} f(x + t_1)$.

Deformation Quantization

Grönewold-van Hove's *"no-go" theorem* prevents standard algebraic quantization: It stops nontrivial linear $O : K[p, q] \to A$, where $K[p, q]$ is the algebra of polynomials in p and q over K and A an associative Lie K-algebra, preserving the Poisson bracket, the identity, and the squares of p and q. This is because $\{f, g\}$ lowers the order of g if f has order $o_f < 2$ (in p and q), does not change order if $o_f = 2$, and raises the order of g if $o_f > 2$, unless g is sent to zero. $K_{\leq 2}[p, q]$ is a maximal subalgebra, so A is isomorpic to $K[p, q]$ with O as the isomorphism (if $K_{\leq 2}[p, q]$ is preserved) or O is zero.

Deformation quantization evades the no-go theorem, replacing the standard product by an associative but non-commuting $\star$, with $f * g = \sum\limits_{k \geq 0} \mu_k(f, g)\hbar^k$, $\mu_0(f, g) = fg$, and $\sum\limits_{0 \leq i \leq n} \mu_{n-i}(\mu_i(f, g), h)) - \mu_{n-i}(f, \mu_i(g, h)) = 0$, and all μ_k are bidifferential operators. The deformation Lie bracket $[\_, \_]$, with $[f, g] = \sum\limits_{k \geq 1} T_k(f, g)\hbar^k = f \star g - g \star f$ such that $T_1(f, g) = \{f, g\}$ (the standard Poisson bracket), the sum over even permutations of $(f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}) = (f_i, f_j, f_k)$ of $\sum\limits_{0 < l < n} T_{n-l}(T_l(f_i, f_j), f_k) = 0$, and all T_k are bidifferential operators. The *Moyal product* $(f \star_M g)(x) = e^{-i\hbar\alpha^{ij}\partial_{x^i}\partial_{y^j}} f(x)g(y)|_{y=x}$, where $\alpha^{ij} = -\alpha^{ji}$. One deformation quantization is the *Weyl quantization* $O^W : C^\infty(\mathbb{R}^n) \to \mathcal{B}(\mathcal{H})$. It starts by asserting for all $f(q, p)$, $f(q, p) \mapsto \hat{f}(q, p) = f(\hat{q}, \hat{p}) = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{4n}} f(q, p) e^{-i(\xi\cdot(q-\hat{q}) + y\cdot(p-\hat{p}))/\hbar} d^n\xi \, d^n y \, d^n q \, d^n p$. $p \cdot q$ maps to the Weyl-ordered product $\frac{1}{2}(\hat{p} \cdot \hat{q} + \hat{q} \cdot \hat{p})$, ensuring well-defined-ness. If $O^W(a) = \hat{a}$, $a = \text{symb}(\hat{a})$ is the *Weyl symbol* of $\hat{a}$. Acting with $O^W(f)$ on an arbitrary function ψ, $(O^W(f)\psi)(z) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} h(p, \frac{x+x'}{2}) e^{-i(x-x')\cdot p/\hbar} \psi(x') d^n p \, d^n x'$ gives the standard form of the Weyl product. This ultimately allows $\text{symb}(\hat{f}\hat{g})$ to be calculated as $f \star_M g$, making $\text{symb}(i\hbar[\hat{f}, \hat{g}]) = \{f, g\}$. If a product $* = \star_M + \star'$ has the property that $O^W(f * g - g * f) = i\hbar[f, g] \; \forall f, g$ has $i\hbar[\hat{f}, \hat{g}] = O^W(f \star_M g - g \star_M f) + O^W(f \star' g - g \star' f)$, so $O^W(f \star' g - g \star' f) = 0$, making $\star'$ a symmetric product $\odot$ by injectivity.

Conclusion and Directions for Future Research

Deformation quantization connects the familiar properties of classical mechanics with the structure of quantum mechanics, and a complete approach to quantization promises to improve the understanding of both. For instance, the formula $* = \star_M + \odot$ may be described as an endpoint of a homotopy $\star_M + t\odot \colon \star_M \to *$ in the space of binary functions on the algebra of observables on T^*Q, a symplectic Poisson manifold. Then, construct the following bundle $\text{Alt}^2(T(T^*Q)) \oplus \text{Sym}^2(T(T^*Q))$, so $t^{ij} = \pi^{ij} + \alpha s^{ij} \in \text{Alt}^2(T(T^*Q)) \oplus \text{Sym}^2(T(T^*Q))$, with $\pi^{ij} \in \text{Alt}^2(T(T^*Q))$ and $s^{ij} \in \text{Sym}^2(T(T^*Q))$, may be constructed such that $f * g = f \star_M g + f \odot g = fg - i\hbar\pi^{ij}(\partial_i f \partial_j g + \partial_j f \partial_i g)/2 + \alpha\hbar s^{ij}(\partial_i f \partial_j g + \partial_j f \partial_i g)/2 + \mathcal{O}(\alpha\hbar^2)$ if $f \odot g$ is $\mathcal{O}(\alpha\hbar)$ (where α itself is $\mathcal{O}(\hbar)$). This is a weak coupling to gravity if s^{ij} is taken as the metric g^{ij}, and exploring geometric properties like the connection and curvature of $\text{Alt}^2(T(T^*Q)) \oplus \text{Sym}^2(T(T^*Q))$ is an interesting area for further study. Also, the representation theory of the Schrödinger representation of the Heisenberg group and deformation quantization performed here could be generalized further.

Notes

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Acknowledgments

The authors would like to acknowledge Professor Brianna Mount for organizing the REU where this research was done, as well as the National Science Foundation (REU Award Number 2150517), Black Hills State University, and the Sanford Underground Research Facility for funding and supporting this research.